

# Generation of polycyclic groups

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To Dan Segal on occasion of his 60-th birthday

## Abstract

We give a new and self-contained proof of a theorem of Linnell and Warhurst that  $d(G) - d(\widehat{G}) \leq 1$  for virtually polycyclic groups  $G$ . We also give a simple sufficient condition for equality  $d(G) = d(\widehat{G})$  when  $G$  is virtually abelian.

## Introduction

Let  $G$  be a finitely generated residually finite group. By  $d(G)$  we denote the minimal size of a generating set for  $G$ , and by  $d(\widehat{G})$  the minimal size of a generating set for the profinite completion  $\widehat{G}$  of  $G$ . In other words

$$d(\widehat{G}) = \max \{d(G/N) \mid N \triangleleft G, G/N < \infty\}.$$

Polycyclic groups are one of the best understood class of groups. For example most of the decision problems are decidable in this class, see [7].

It seems surprising therefore that it is still an open problem whether there exists an algorithm which finds  $d(G)$  for any polycyclic group  $G$  (given by say a set of generators and relations). This is unknown even in the case when  $G$  is virtually abelian.

It is obvious that  $d(G) \geq d(\widehat{G})$  and when there is equality both the value of  $d(G)$  and a minimal generating set for  $G$  can indeed be found algorithmically. (Say by enumerating both the finite images and all possibilities for generating sets for  $G$ ).

In general  $d(G) - d(\widehat{G})$  can be arbitrarily large even for metabelian groups  $G$ , see [4]. In fact Wise [9] has proved that there exist groups  $G$  with arbitrarily large  $d(G)$  while  $d(\widehat{G}) = 3$ .

Fortunately for polycyclic groups the situation is not that bad. In [3] Linnell and Warhurst proved the following theorem using methods from commutative algebra and lattices over orders.

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**Theorem 1.** *Let  $G$  be a virtually polycyclic group. Then  $d(G) \leq d(\widehat{G}) + 1$ .*

Note that the inequality is sharp even for virtually abelian groups: many examples with  $d(G) = d(\widehat{G}) + 1$  are constructed in [5].

In this note we give an alternative proof of Theorem 1. While not claiming anything new we believe that our argument is much simpler than the original one in [3]. Moreover our result gives some sufficient condition when  $d(G) = d(\widehat{G})$  which can be verified quite easily in the case when  $G$  is virtually abelian.

**Theorem 2.** *Let  $G$  be a group with normal finitely generated abelian subgroup  $U$  such that  $G/U$  is finite. Let  $d_p(G) = d(G/U^p)$  for any prime  $p$ . Then*

$$\alpha = d(\widehat{G}) \leq d(G) \leq k := \max\{\alpha, \beta + 1\} \leq d(\widehat{G}) + 1, \quad (1)$$

where  $\alpha = \max_p d_p(G)$  and  $\beta = \min_p d_p(G)$ .

Moreover for any integer  $N \in \mathbb{N}$  there exists a generating set  $S$  for  $G$  of size  $k$ , such that the first  $d(\widehat{G})$  elements generate a subgroup of index co-prime to  $N$ . The same result holds for finitely generated virtually nilpotent groups.

In particular if there are two primes  $p$  and  $q$  such that  $d_p(G) \neq d_q(G)$  then  $d(G) = d(\widehat{G})$ .

Note that Theorem 2 easily implies a weaker version of Theorem 1, namely that  $d(G) \leq d(\widehat{G}) + 2$ , however obtaining the right bound  $d(\widehat{G}) + 1$  is harder. For that we need a general and somewhat technical result (Theorem 7 on lifting generators) proved in Section 1. The proofs of Theorems 2 and 1 are then immediate and are given in Section 2.

## Notation

For elements  $a, b \in G$  in a group  $G$  the commutator  $[a, b]$  of  $a$  and  $b$  is  $aba^{-1}b^{-1}$ .

## 1 Lifting generators

In this section we shall prove a general result which under certain condition produces a generating set of a group  $G$  starting from a generating set of some quotient  $G/V$  of  $G$ . The reason for stating it in such generality is because in the next section we shall apply it in two settings: when  $G$  is a virtually abelian group and then when  $G$  is virtually metabelian.

First recall the following result by Gaschütz ([1]).

**Theorem 3.** *Let  $G$  be a finite group with a normal subgroup  $N$ . Let  $d \geq d(G)$  and let  $a_1, \dots, a_d$  be any  $d$  elements which generate  $G \bmod N$ , i.e.,  $G = N\langle a_1, \dots, a_d \rangle$ . Then we can find elements  $g_i \in a_i N$  ( $i = 1, 2, \dots, d$ ) such that  $G = \langle g_1, \dots, g_d \rangle$ .*

**Definition 4.** Let  $G$  be a group and  $p$  be a prime. A normal subgroup  $L$  of finite index in  $G$  is  $p$ -good if for any subgroup  $H \leq G$  with  $HL = G$  we have that  $[G : H]$  is finite and coprime to  $p$ .

It is not difficult to see that  $p$ -good subgroups exists for any virtually polycyclic group  $G$ , see Lemma 11 below.

Now let  $G$  be a finitely generated group with an abelian normal subgroup  $V$  of finite rank. Suppose that for every prime number  $p$  we have chosen a  $p$ -good subgroup  $G_p$  of  $G$  such that  $G_p \geq V^p$ .

**Definition 5.** We say that  $h_1, \dots, h_k$  generate  $G$  mod  $p$  if  $\langle h_1, \dots, h_k \rangle G_p = G$ . Let  $d_p(G) = d(G/G_p)$  denote the minimal size of a set of generators for  $G$  mod  $p$ .

**Definition 6.** Let  $w = w(x_1, \dots, x_n)$  be a group word (element in the free group  $F$ ). The Fox derivatives  $\frac{\partial w}{\partial x_i}$  are elements in the group ring  $\mathbb{Z}[F]$ , which are defined by  $\frac{\partial x_j}{\partial x_i} = \delta_{ij}$  and

$$\frac{\partial uv}{\partial x_i} = \frac{\partial u}{\partial x_i} + u \frac{\partial v}{\partial x_i}.$$

Let  $G$  be a group and  $V$  be a  $G$ -module. For any  $n$ -tuple  $\underline{g} = (g_1, \dots, g_n) \in G^n$  the Fox derivative  $\frac{\partial w}{\partial x_i}$  naturally defines a map  $\frac{\partial w}{\partial x_i}(\underline{g}) : V \rightarrow V$ .

An equivalent way to define this map is the following: Let  $\Gamma$  be an extension of  $G$  by the abelian group  $V$  then

$$\frac{\partial w}{\partial x_i}(\underline{g})(a) = w(\gamma_1, \dots, \gamma_{i-1}, a\gamma_i, \gamma_{i+1}, \dots, \gamma_n) \cdot w(\gamma_1, \dots, \gamma_n)^{-1}$$

for any lifts  $\gamma_i \in \Gamma$  of  $g_i \in G$ .

**Theorem 7.** Let  $\underline{\gamma} = (\gamma_1, \dots, \gamma_k)$  be a set of elements in  $G$  which generate  $G/V$ . Suppose that  $w(x_1, \dots, x_k)$  is a word such that  $w(\underline{\gamma}) \in V$ .

Assume that

1. The image of the map  $\pi : V \rightarrow V$  defined by  $\pi(v) = \frac{\partial w}{\partial x_k}(\underline{\gamma}) \circ v$  has finite index  $M$  in  $V$ .
2. For any choice of elements  $g_1 \in \gamma_1 V, \dots, g_{k-1} \in \gamma_{k-1} V$  the group  $\langle g_1, \dots, g_{k-1} \rangle$  generated by them has finite index in  $G$ .
3. We have that  $d_p(G) \leq k$  for any prime  $p$ .

Then there exist lifts  $g_1, \dots, g_k$  of  $\gamma_1, \dots, \gamma_k$  (i.e. such that  $g_i \in \gamma_i V$ ) which generate  $G$ .

Moreover there is an algorithm for finding  $g_i$  from the  $\gamma_i$  (provided all the objects from conditions 1,2,3 above are computable).

*Proof.* We say that the element  $g_i$  is a lift of  $\gamma_i$  whenever  $g_i \in \gamma_i V$ . Note that the Fox derivative  $\frac{\partial w}{\partial x_i}(\underline{g}) = \frac{\partial w}{\partial x_i}(\underline{\gamma})$  does not depend on the choice of lifts  $\underline{g} = (g_1, \dots, g_k)$  of  $\underline{\gamma} = (\gamma_1, \dots, \gamma_k)$ .

For each  $i = 0, 1, \dots, k$  let  $Q(i)$  be the following statement.

**Q(i) :** *There exist lifts  $S_i = \{g_1, \dots, g_i\}$  of  $\gamma_1, \dots, \gamma_i$  and a finite set of prime numbers  $P_i$  with the following property.*

- *For each prime  $p \in P_i$  there exist lifts  $g_j^{(p)} \in \gamma_j V$ ,  $j = i+1, \dots, k$  such that  $S_i \cup \{g_j^{(p)}\}_{j>i}$  are  $k$  generators for  $G \bmod p$ .*
- *For each prime  $p \notin P_i$ , there exist lifts  $g_j^{(p)}$ ,  $j = i+1, \dots, k-1$  such that for any lift  $g_k^{(p)} \in \gamma_k V$  we have that  $S_i \cup \{g_j^{(p)}\}_{j>i}$  are  $k$  generators for  $G \bmod p$ .*

The proof of  $Q(i)$  is by induction on  $i$ .

The base case  $i = 0$  is proved as follows: Take  $S_0 = \emptyset$  and choose any lifts  $\gamma_1, \dots, \gamma_{k-1}$  of the  $\gamma_j$ . They generate a subgroup of finite index  $L$  in  $G$ , therefore for any  $p \nmid L$  and any lift  $g_k \in \gamma_k V$  the elements  $\{g_1, \dots, g_k\}$  generate  $G \bmod p$ .

Define  $P_0$  to be the set consisting of all primes which divide  $L$  or  $M$ . We only have to show that for all  $p \in P_0$  there exist lifts  $g_1^{(p)}, \dots, g_k^{(p)}$ , which generate  $G \bmod p$ . Consider the  $k$  images  $\bar{\gamma}_j$  of  $\gamma_1, \dots, \gamma_k$  in  $G/VG_p$ . They generate  $G/VG_p$  (since  $\gamma_j$  generate  $G/V$ ) and also we know that  $G/G_p$  is  $k$ -generated. By Gaschutz Theorem we can find elements  $g_j^{(p)} \in \gamma_j VG_p$  which generate  $G/G_p$ . These can be further adjusted by elements from  $G_p$  so that  $g_j^{(p)} \gamma_j^{-1} \in V$ . This proves the base case  $i = 0$  of the induction.

Suppose that we have already found  $S_i$  and  $P_i$ . By the Chinese Remainder Theorem there exists a lift  $g_{i+1}$  such that  $g_{i+1} = g_{i+1}^{(p)} \pmod{G_p}$  for all  $p \in P_i$ .

Choose any lifts  $g_{i+2}, \dots, g_{k-1}$ . By one of the assumptions the group  $H$  generated by  $\{g_i\}_{i=1}^{k-1}$  is of finite index  $N_i$  in  $G$ . Denote  $P_{i+1} = \{p \mid p \text{ divides } N_i\} \cup P_i$ .

We want to show that  $S_{i+1} = S_i \cup \{g_{i+1}\}$  and the set  $P_{i+1}$  satisfy the induction hypothesis.

It is very easy to check that the second part of the induction hypothesis is satisfied for this definition of the set  $P_{i+1}$  (just choose  $g_j^{(p)} = g_j$  for  $j = i+2, \dots, k-1$ ). It remains to show that for all primes  $p \in P_{i+1}$  the first condition is satisfied. This is clearly the case if  $p \in P_i$ . Let  $p \notin P_i$  then by the induction assumption there exist lifts  $\{g_j^{(p)}\}_{j=i}^{k-1}$  which together with  $S_i$  and any lift  $g_k^{(p)} \in \gamma_k V$  generate  $G \bmod p$ . We will show that we can choose a lift  $g_k$  of  $\gamma_k$  such that the group  $L$  generated by  $S_{i+1}$ ,  $\{g_j^{(p)}\}_{j=i+1}^{k-1}$  and  $g_k$  contains an element  $u \equiv g_{i+1}^{-1} g_{i+1}^{(p)} \pmod{V^p}$ , which implies that these elements generate  $G \bmod p$ .

The key observation here is that for  $x \in V$  the element  $w(g_1^{(p)}, \dots, g_{k-1}^{(p)}, xg_k^{(p)})$  is equal to

$$\left( \frac{\partial w}{\partial x_k}(\underline{\gamma}) \cdot x \right) w(g_1^{(p)}, \dots, g_{k-1}^{(p)}, g_k^{(p)}) = \pi'(x) \in V$$

which by one of the assumptions takes any value in  $V/V^p$  as  $x$  ranges over  $V/V^p$  (since  $p$  does not divide the index  $M$  of the image of  $\frac{\partial w}{\partial x_k}(\underline{\gamma})$  in  $V$ ). So we can indeed find  $x \in V$  such that the element  $\pi'(x) \in V$  satisfies  $\pi'(x) \equiv g_{i+1}^{-1}g_{i+1}^{(p)} \pmod{V^p}$ .

This shows that there exist lifts  $g_k$  which generate  $G \pmod{p}$  which completes the induction step.

The statement  $Q(k)$  gives a set  $S_k$  which generates  $G \pmod{p}$  for any prime  $p$  and therefore  $\langle S_k \rangle = G$ .

It is clear that this argument in fact produces an algorithm for finding the set  $S_k$  in a very efficient way, of course provided the various subgroup indices, words and maps  $G \rightarrow G/G_p$  involved in the induction are computable. Theorem 7 is proved.  $\square$

*Remark 8.* A slight modification of the proof gives that for any finite set  $P$  of primes such that  $d_p(G) < k$ , we can find lifts  $g_1, \dots, g_{k-1}$  which together with any lift of  $\gamma_k$  generate a subgroup of index not divisible by any prime in  $P$ .

*Remark 9.* If  $\gamma_k = e$  then we can take the word  $w = x_k$ . Its Fox derivative is  $\frac{\partial w}{\partial x_k} = e$  and defines the identity map from  $V$  to  $V$ , which is clearly surjective.

*Remark 10.* If we replace the assumption that  $V$  is abelian with  $V$  nilpotent, then all results remain valid, since a set generates a nilpotent group if and only if it generates the abelianization of the group.

## 2 Applications of Theorem 7

### 2.1 Proof of Theorem 2

*Proof.* Clearly  $d(\widehat{G}) = \alpha$ . Let  $k := \max\{\beta + 1, \alpha\}$ .

Take  $V := U^q$  where  $q$  is a prime such that  $\beta = d_q(G)$ . Then  $V$  is  $q$ -good, i.e., any collection of elements which generates  $G/V$  generates a subgroup of finite index (coprime to  $q$ ) in  $G$ .

Take elements  $\gamma_1, \dots, \gamma_\beta$  which generate  $G/V$ . Set  $\gamma_i = 1$  for  $j = \beta + 1, \dots, k$ . It is easy to see that the group  $G$ , subgroups  $V$ ,  $G_p = V^p$  (for any prime  $p$ ), the elements  $\gamma_i$  above, and the word  $w = x_k$  satisfy the conditions of Theorem 7. We conclude that  $G$  can be generated by some lifts of  $\gamma_1, \dots, \gamma_k$  and so  $d(G) \leq k$  as claimed.

For the second part of the theorem start with  $V = U^N$  instead and with any generating set  $\gamma_1, \dots, \gamma_s$  for  $G/V$  and again take  $\gamma_j = 1$ ,  $s < j \leq k$ . The rest of the argument is similar.  $\square$

## 2.2 Proof of Theorem 1

We begin with the following straightforward

**Lemma 11.** *If  $G$  is a virtually polycyclic group and  $p$  is a prime then  $G$  has a  $p$ -good subgroup  $L$ .*

*Proof.* We use induction on the Hirsch length  $h(G)$  of  $G$ . When  $h(G) = 0$  then  $G$  is finite and we can simply take  $L = 1$ . Suppose that the Lemma has been proved for all groups of Hirsch length less than  $h > 0$ . Consider a virtually polycyclic group  $G$  with  $h(G) = h$ . Since  $G$  is infinite it has an infinite normal abelian subgroup  $N$ . Let  $\overline{G} = G/N^p$ . Clearly  $h(\overline{G}) < h$  and so  $\overline{G}$  has a  $p$ -good subgroup, say  $\overline{L} = L/N^p$  for some normal subgroup  $L$  containing  $N^p$ .

Then  $L$  is  $p$ -good for  $G$ : Suppose  $H \leq G$  with  $HL = G$ . Then  $\overline{HL} = \overline{G}$  where  $\overline{H} = HN^p/N^p$  and so the index  $[G : HN^p]$  is finite and coprime to  $p$ . Therefore  $HN = HN^p$  (since  $N/N^p$  is a power of  $p$  while  $[HN : HN^p]$  must be coprime to  $p$ ) and so  $H_1N^p = N$  where  $H_1 = H \cap N$ . The last equality implies that  $[N : H_1]$  is finite and coprime to  $p$  and hence so is  $[G : H] = [G : NH][N : H_1]$ .

Notice that the  $p$ -good subgroup  $L$  we found contains  $N^p$ .  $\square$

Lemma 11 raises the following natural

*Problem 1.* Which groups  $G$  possess a subgroup  $N$  of finite index such that  $NH = G$  for a subgroup  $H < G$  implies that  $[G : H] < \infty$ ?

The purpose of the following three Lemmas is to ensure the existence of suitable word  $w$ , subgroup  $V$  and elements  $g_1, \dots, g_k$  in a virtually metabelian and polycyclic group  $G$  which meet the conditions of Theorem 7.

**Lemma 12.** *Let  $G$  be a virtually metabelian and polycyclic group. Then there exist normal subgroups  $G_0 \triangleright V$  of  $G$  such that*

1.  $G/G_0$  is finite,  $V$  is a torsion free abelian group,
2. if  $H$  is a subgroup of  $G$  such that  $HG_0 = G$  then  $H$  is of finite index in  $G$ ,
3.  $G_0/V$  is a nilpotent group which acts commutatively on  $V$ , i.e.,  $G_0/C_{G_0}(V)$  is abelian, and
4.  $\mathbb{Q} \otimes V$  is a perfect  $\mathbb{Q}[G_0/V]$  module, i.e.,  $(G_0 - 1) \cdot V$  has finite index in  $V$ .

*Proof.* Let  $A < B$  be normal subgroups of  $G$  such that  $A$  and  $B/A$  are torsion free abelian and  $[G : B]$  is finite. Let  $L$  be a  $p$ -good subgroup of  $G$  for some prime  $p$  and take  $G_0 = B \cap L$ . Then  $G_0$  also a  $p$ -good subgroup of  $G$  and item 2 follows.

Let  $W = A \cap G_0$ . We have that  $W$  and  $G_0/W$  are torsion free abelian groups and  $W$  is a module for  $\tilde{G}_0 = G_0/W$ . Consider the chain of submodules  $W \geq (G_0 - 1)W \geq (G_0 - 1)^2W \geq \dots$ . This is a chain of subgroups of the finitely generated abelian group  $W$ , so let  $V$  be the first module in that series

such that  $(G_0 - 1)V$  has finite index in  $V$ . Clearly  $G_0/V$  is a nilpotent group (since  $G_0$  acts nilpotently on  $W/V$ ). Item 4 is clear since  $[V : (G_0 - 1)V]$  is finite while item 3 follows since  $C_{G_0}(V) \geq W$  and  $G_0/W$  is abelian.  $\square$

**Lemma 13.** *Let  $\Gamma_0$  be a finitely generated torsion free abelian group and let  $V$  be a finitely generated torsion free  $\mathbb{Z}[\Gamma_0]$  module such that  $V_{\mathbb{Q}} = V \otimes \mathbb{Q}$  is a perfect  $\mathbb{Q}[\Gamma_0]$ -module. Then there exists an integer  $N$  such that for any subgroup  $\Gamma < \Gamma_0$  of index co-prime to  $N$  we have that  $V_{\mathbb{Q}}$  is a perfect  $\mathbb{Q}[\Gamma]$ -module.*

*Further when this happens then we can find an integer  $M \in \mathbb{N}$  (depending on  $\Gamma$ ), integers  $s_i$  and group elements  $h_i \in \Gamma$  such that*

$$\sum s_i(1 - h_i) = M \cdot id$$

*as operators on  $V$ .*

*Proof.* Let  $\chi$  be an irreducible character (over  $\mathbb{C}$ ) of  $\Gamma_0$ . We will call  $\chi$  a character of finite order if all values of  $\chi$  are roots of 1, in this case the order of  $\chi$  is the least integer  $n$  such that all its values are  $n$ -th roots of 1.

Since  $V$  is a perfect  $\mathbb{Q}[\Gamma_0]$  module it does not contain a trivial submodule. Let  $N$  be the gcd of all orders of irreducible characters which appears in  $\mathbb{C}V$ . If  $\Gamma \leq \Gamma_0$  is a subgroup of index co-prime to  $N$  then the restriction of any irreducible characters in  $V$  to  $\Gamma$  is non trivial. Therefore  $\mathbb{C}V$  is a perfect  $\mathbb{C}[\Gamma]$  module, which implies that  $V_{\mathbb{Q}} = \mathbb{Q} \otimes V$  is perfect  $\mathbb{Q}[\Gamma]$  module.

For the second part, let  $I$  be the augmentation ideal of  $\mathbb{Q}\Gamma$ . Since  $V_{\mathbb{Q}} = IV_{\mathbb{Q}}$  and  $V_{\mathbb{Q}}$  is a finite dimensional vector space over  $\mathbb{Q}$  we have by Nakayama's lemma that  $id + T$  annihilates  $V_{\mathbb{Q}}$  for some  $T \in I$ .

Expressing  $T$  in the basis of  $I$  and clearing the common denominator  $M$  of the rational coefficients gives the integers  $s_i$  and the elements  $h_i \in G$ .  $\square$

**Lemma 14.** *Let  $\Gamma$  be a group with an abelian normal subgroup  $V$  such that  $[\Gamma', V] = 1$  and  $\Gamma/V$  is nilpotent of class  $d$ . Let  $g_1, \dots, g_{k-1} \in \Gamma$ .*

*For any integers  $t \in \mathbb{N}, s_i$  and group elements  $h_i \in \langle g_1, \dots, g_{k-1} \rangle$  ( $i = 1, \dots, t$ ) there exists a word  $w$  on  $x_1, \dots, x_k$  such that*

- $w(g_1, \dots, g_{k-1}, g) \in V$  for any  $g \in \Gamma$
- The action of Fox derivative  $\frac{\partial w}{\partial x_k}(g_1, \dots, g_{k-1}, g)$  on  $x \in V$  is given by

$$x \mapsto \left( \sum_{i=1}^t s_i(1 - h_i) \right)^d \cdot x.$$

*Proof.* Consider the word  $w'(g_1, \dots, g_{k-1}, g) = \prod [g, h_i]^{s_i}$  where  $h_i$  are expressed as words on  $g_1, \dots, g_{k-1}$ . A direct computation gives that the Fox derivative  $\frac{\partial w}{\partial x_k}$  at  $(g_1, \dots, g_{k-1}, g)$  with respect to the last variable acts on  $V$  as multiplication by  $\sum s_i(1 - h_i) \in \mathbb{Z}[\Gamma/V]$  (use that  $[ag, h_i] = a[g, h_i](h_i a)^{-1} = {}^{(1-h_i)}a \cdot [g, h_i]$  and each  $[g, h_i]$  acts trivially on  $V$ ).

Iterating the map  $g \rightarrow w'(g_1, \dots, g_{k-1}, g)$   $d$  times gives a word  $w$ :

$$w(x_1, \dots, x_k) = w' \left( x_1, \dots, x_{k-1}, w' \left( x_1, \dots, x_{k-1}, w(\dots w'(x_1, \dots, x_{k-1}, x_k) \dots) \right) \right).$$

The Fox derivative of  $w$  with respect to  $x_k$  is  $(\sum s_i(1 - h_i))^d$ , because substitution of words corresponds to multiplication of Fox derivatives. The word  $w$  always evaluates to one on  $\Gamma/V$ , because the group  $\Gamma/V$  is nilpotent of class  $d$ .  $\square$

We now have all the ingredients to prove Theorem 1. It will follow from the corresponding result for metabelian groups:

**Theorem 15.** *Let  $G$  be a virtually metabelian polycyclic group. Then*

$$d(\widehat{G}) \leq d(G) \leq d(\widehat{G}) + 1.$$

*Proof.* Let  $k = d(\widehat{G}) + 1$ . Let  $G_0$  and  $V$  are the subgroups provided by Lemma 12. Now Lemma 13 applied to the group  $G_0/[G_0, G_0]V$  acting on  $V$ , gives us an integer  $N$  such that any subgroup of index co-prime to  $N$  in  $G_0/V$  acts perfectly on  $V$  (as a rational module).

By Lemma 2 there exists a generating set  $S = \{\gamma_1, \dots, \gamma_k\}$  of  $G/V$  such that  $\gamma_k \in G_0$  and the subgroup  $\Gamma = \langle \gamma_1, \dots, \gamma_{k-1} \rangle V \cap G_0$  has index  $[G_0 : \Gamma]$  co-prime to  $N$ . Therefore  $V$  is a perfect rational  $\mathbb{Q}[\Gamma/V]$ -module and for some integer  $M$  and element  $T$  in the augmentation ideal of  $\mathbb{Z}[\Gamma/V]$  we have that  $T$  acts on  $V$  as multiplication by  $M$ .

For each prime  $p$  pick a  $p$ -good subgroup  $G_p$  of  $G$  containing  $V^p$ . (In fact by replacing  $G_p$  with a normal subgroup of  $G$  of finite index we may even assume  $G_p \cap V = V^p$ .)

Now apply Lemma 14 to  $\Gamma$  and  $V$  with  $\gamma_i = g_i$ , ( $i = 1, \dots, k-1$ ),  $g = \gamma_k$  and  $s_i \in \mathbb{N}$ ,  $h_i \in \langle \gamma_1, \dots, \gamma_{k-1} \rangle$  chosen so that  $T \equiv \sum_i s_i(1 - h_i)$  in  $\mathbb{Z}[\Gamma/V]$ . We conclude that there is a word  $w(x_1, \dots, x_k)$  such that  $w(\underline{\gamma}) \in V$  and  $\frac{\partial w}{\partial x_k}(\underline{\gamma})$  acts on  $V$  as multiplication by  $M^d$  where  $d$  is the nilpotency class of  $\Gamma/V$ .

We can now apply Theorem 7 to  $G$  with these choices of  $w, V, G_p$  and  $\gamma_j$ . The conditions 1,2 and 3 are satisfied by the construction of the subgroups  $G_0$  and  $V$ , the word  $w$  and the definition of the number  $k$ . So by Theorem 7 we can find lifts  $a_i \in \gamma_i V$  such that  $G = \langle a_1, \dots, a_k \rangle$ . Theorem 15 is proved.  $\square$

*Remark 16.* If we have that  $d(G/V) < d(\widehat{G})$  then the argument above gives that  $d(G) = d(\widehat{G})$ .

*Proof of Theorem 1.* In general a virtually polycyclic group  $G$  is virtually nilpotent by abelian, i.e., it has normal subgroups  $G_1 > G_2$  such that  $G/G_1$  is finite,  $G_1/G_2$  is abelian while  $G_2$  is nilpotent. (See Theorem 2, Chapter 2 in [6]).

Now every group which generates  $H = G/G'_2$  generates  $G$  and so we have  $d(G) = d(H)$ ,  $d(\widehat{G}) = d(\widehat{H})$ . Thus Theorem 1 becomes a corollary of Theorem



15. Moreover its proof gives an efficient algorithm for generating a polycyclic group  $G$  with  $d(\widehat{G}) + 1$  elements, even with  $d(\widehat{G})$  elements if the condition of Remark 16 holds.  $\square$

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